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FOREWORD

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This technical report has been reviewed and is approved.

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ABSTRACT

An operationally simple strength criterion for anisotropic materials is developed from a scalar function of two strength tensors. Differing from existing quadratic approximations of failure surfaces, the present theory satisfies the invariant requirements of coordinate transformation, treats interaction terms as independent components, takes into account the difference in strengths due to positive and negative stresses, and can be specialized to account for different material symmetries, multi-dimensional space, and multi-axial stresses. The measured off-axis uniaxial and pure shear data are shown to be in good agreement with the predicted values based on the present theory.

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Nomenclature

F_i	Strength tensor of the 2nd rank
F_{ij}	Strength tensor of the 4th rank
m,n	cos θ, sin θ
P, P'	Plane hydrostatic tensile and compressive stresses
Q, Q'	Positive and negative shear strengths in 2-3 plane
R, R'	Positive and negative shear strengths in 3-1 plane
S, S'	Positive and negative shear strengths in 1-2 plane
U, U'	45-degree tensile and compressive strengths in 1-2 plane
V, V'	45-degree positive and negative shear strengths in 1-2 plane
X, X'	Tensile and compressive strengths along the 1-axis
Y, Y'	Tensile and compressive strengths along the 2-axis
Z, Z'	Tensile and compressive strengths along the 3-axis
T	Combined tension and torsion strength
T_i	Transformation equations for F_i (Table I)
U_i, V_i, W_i	Transformation equations for F_{ij} (Table II)
σ_i	Stress components
ε_i	Strain components

ILLUSTRATIONS

Introduction

For the purpose of material characterization and design, an operationally simple strength criterion for filamentary composites is essential. Strength is an elusive and ambiguous term. It covers many aspects associated with the failures of materials such as fracture, fatigue and creep, under quasi-static or dynamic loading, exposed to inert or corrosive environments, subjected to uni- or multi-axial stresses, in 2 or 3-dimensional geometric configurations, etc. Failures of composites are further complicated by a multitude of independent and interacting mechanisms which include filament breaks and micro-buckling, delamination, dewetting, matrix cavitation and crack propagation. An operationally simple strength criterion cannot possibly explain the actual mechanisms of failures. It is intended only as a useful tool for materials characterization, which determines how many independent strength components exist and how they are measured; and for design which requires a relatively simple method of estimating the load-carrying capacity of a structure.

There have been numerous strength criteria in existence and additional ones are frequently being proposed. In the ASTM's Composite Materials: Testing and Design,[1] various strength criteria are used or alluded to by nearly one half of all the contributors. Nearly all of them agree with one another with reference to the principal strengths; i.e., those uniaxial and pure

shear test data measured along material symmetry axes. Those strengths are the intercepts of the failure surface with the coordinate axes in the stress-space. The disagreements among existing criteria usually occur in the combined-stress state; i.e., in the space away from the coordinate axes of the failure surface. Since reliable experimental data in the combined-stress state were almost nonexistent until very recently, no serious attempt to challenge the validity of various failure criteria has been undertaken. Recent achievements in the automation of testing in the combined-stress state and in the preparation of unidirectional and laminated tubular test specimens of high quality have resulted in meaningful property data in the combined-stress state for the first time in the Western World. It is, therefore, timely to examine the validity and utility of existing strength criteria.

A number of Russian workers including Gol'denblat[2] have tried to describe strength criteria using strength tensors. Their criteria were not operationally simple, and did not investigate problems of material symmetries, coordinate transformations, and the associated invariants. It is also a common Russian practice to apply the failure criteria to the entire laminates instead of the unidirectional constituent layers. Such practice imposes a less severe test on the validity of the criteria because laminates have much weaker anisotropy than unidirectional layers. Available Russian data do indicate that failure surfaces for many weakly anisotropic materials can be

approximated in stress-space by ellipsoids rotated and displaced from their principal axes. The popular failure criteria in the Western World have been strongly influenced by Hill[3] who developed his strength theory for specially orthotropic materials by "generalizing" the Mises criterion for plastically incompressible isotropic materials. As we will attempt to show later in this paper, such generalization is a risky endeavor and should be avoided. Other workers in the West have employed the maximum stress and maximum strain theories. Although the approaches are operationally simpler than those proposed by the Russian workers, they are in effect curve-fitting schemes and have no analytic foundation. It is our intent to develop an operationally simple failure criterion from strength tensors. Obvious advantages of this approach are that the invariance, the transformation properties, the material symmetries, and commonality between various levels of spatial dimensions and multi-axial stresses can all be deduced from a general theory with established properties of tensors and without additional assumptions. Unlike arbitrary curve-fitting schemes which lack analytic foundation, our approach will have built-in generality and internal consistency.

Basic Assumption

The basic assumption of our strength criterion is that there exists a failure surface in the stress-space in the

following scalar form:

$$f(\sigma_k) = F_i\sigma_i + F_{ij}\sigma_i\sigma_j = 1 \tag{1a}$$

where the contracted notation is used; and i, j, k = 1,...6; F_i and F_{ij} are strength tensors of the second and fourth rank, respectively. Since the use of contracted notation does not follow a universal pattern, we will show in detail our usage in Appendix I. Equation (1a) in expanded or long-hand form is:

$$\begin{aligned}
&F_1\sigma_1 + F_2\sigma_2 + F_3\sigma_3 + F_4\sigma_4 + F_5\sigma_5 + F_6\sigma_6 \\
&\quad + F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + 2F_{13}\sigma_1\sigma_3 + 2F_{14}\sigma_1\sigma_4 + 2F_{15}\sigma_1\sigma_5 + 2F_{16}\sigma_1\sigma_6 \\
&\qquad + F_{22}\sigma_2^2 + 2F_{23}\sigma_2\sigma_3 + 2F_{24}\sigma_2\sigma_4 + 2F_{25}\sigma_2\sigma_5 + 2F_{26}\sigma_2\sigma_6 \\
&\qquad\quad + F_{33}\sigma_3^2 + 2F_{34}\sigma_3\sigma_4 + 2F_{35}\sigma_3\sigma_5 + 2F_{36}\sigma_3\sigma_6 \\
&\qquad\qquad + F_{44}\sigma_4^2 + 2F_{45}\sigma_4\sigma_5 + 2F_{46}\sigma_4\sigma_6 \\
&\qquad\qquad\quad + F_{55}\sigma_5^2 + 2F_{56}\sigma_5\sigma_6 \\
&\qquad\qquad\qquad + F_{66}\sigma_6^2 = 1 \ \ldots\ldots\ldots\ (1b)
\end{aligned}$$

The linear term in σ_i takes into account internal stresses which can describe the difference between positive- and negative-stress induced failures. The quadratic terms $\sigma_i\sigma_j$ define an ellipsoid in the stress-space. It is assumed that failure occurs when a stress vector reaches the failure surface. The stress vector cannot extend beyond the failure surface. Inside the surface,

no failure occurs and the material is elastic. The material up to failure is independent of the loading path. Then we know that F_{ij} is symmetric when the strength potential function f in Equation (1) is of class C2; i.e.,

$$F_{ij} = \frac{\partial^2 f}{\partial\sigma_i \partial\sigma_j} = \frac{\partial^2 f}{\partial\sigma_j \partial\sigma_i} = F_{ji} \tag{2}$$

F_i must be symmetric because σ_i is symmetric; this is shown in Appendix I. In our basic assumption in Equation (1), we ignored higher order terms, e.g., term $F_{ijk}\sigma_i\sigma_j\sigma_k$ in the strength criterion is not practical from the operational standpoint because the number of components in a 6th-rank tensor run into the hundreds. In addition, having cubic terms, the failure surface becomes open-ended.

Several features of our proposed strength criterion are as follows:

(1) It is a scalar equation as compared with 6 simultaneous equations required by the maximum stress or maximum strain criterion. Interactions among all stress components are independent material properties; whereas interactions are not permitted in the maximum stress or maximum strain theories. In criteria by quadratic approximations such as Hill's[3], interactions are fixed (not independent).

(2) Since strength components are expressed in tensors, their transformation relations and the associated invariants

are well established. In particular, the transformation relations in terms of multiple angles, similar to those developed for the elastic stiffness[4], are useful tools for the understanding of strength tensors. These relations are shown in Appendix II.

(3) The symmetry properties of the strength tensors and the number of independent and non-zero components can be rigorously described similar to other well-established properties of anisotropic materials, such as the elastic compliance matrix. The number of spatial dimensions and multi-axial stresses are determined by selecting the proper range of the indices among 1 to 6. General anisotropy and 3-dimensional space present no conceptual difficulty.

(4) Knowing the transformation relations in Appendix II. we can readily rotate the material axes from F_i to F_i' and F_{ij} to F_{ij}' in Equation (1), or equivalently rotate in the opposite direction to change the applied stresses from σ_i to σ_i' when we want to study the off-axis or transformed properties. Most existing criteria are limited to specially orthotropic materials. These criteria can only be applied by transforming the external stresses to the material axes. Rotation of the material axes cannot be done because the transformations of the strength criteria are not known.

(5) Being invariant, Equation (1) is valid for all coordinate systems when it is valid for one coordinate system. Such validity holds for curvilinear coordinates as well with only minor adjustments.

(6) Certain stability conditions are incorporated in the strength tensors. The magnitudes of interaction terms are constrained by the following inequality:

$$F_{ii}F_{jj} - F_{ij}^{2} \geq 0 \qquad (3)$$

where repeated indices are NOT summations for this equation; and i, j=1,...,6. F_{ii} is simply one of the diagonal terms. To be physically meaningful, all diagonal terms must be positive; the off-diagonal or interaction terms may be positive or negative depending on the nature of the interaction but their magnitudes are constrained by the inequality in Equation (3). Geometrically, this inequality ensures that our failure surface will intercept each stress axis. The shape of the surface will be ellipsoidal. The failure surface cannot be open-ended like a hyperboloid. Equation (3) makes sure that it will not happen. The same positive definite requirement of F_{ij} is imposed on F_i. The displacements of the ellipsoid due to internal stresses are such that the origin remains inside the ellipsoid.

(7) Finally, Gol'denblat[2] was one of the first to suggest the use of strength tensors and proposed a general theory in the

following form:

$$(F_i\sigma_i)^{\alpha} + (F_{ij}\sigma_i\sigma_j)^{\beta} + (F_{ijk}\sigma_i\sigma_j\sigma_k)^{\gamma} + \ldots = 1 \qquad (4a)$$

He investigated a special case of

$$\alpha = 1, \quad \beta = 1/2, \quad \gamma = -\infty \, . \qquad (4b)$$

The ± sign associated with the square root is awkward. This, however, can be eliminated by simple rearrangement of this special case and we have:

$$2F_i\sigma_i + F_{ij}\sigma_i\sigma_j - (F_i\sigma_i)^2 = 1 \qquad (4c)$$

This relation is also more complicated than Equation (1). The additional term does not introduce any more generality than the linear and quadratic approximation of Equation (1); i.e., there are 6 linear and 21 quadratic terms. We believe that our approximation is operationally simpler and will be investigated in detail in this paper.

Symmetry Properties

The symmetry properties of our strength tensors follow well established patterns of the diffusion[5] and elastic[6] properties of anisotropic materials. For a triclinic material in 3-space:

$$F_i = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \tag{5a}$$

$$F_{ij} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ & F_{22} & F_{23} & F_{24} & F_{25} & F_{26} \\ & & F_{33} & F_{34} & F_{35} & F_{36} \\ & & & F_{44} & F_{45} & F_{46} \\ & & & & F_{55} & F_{56} \\ & & & & & F_{66} \end{bmatrix} \tag{5b}$$

As stated earlier, we are assuming that both strength tensors are symmetric. The number of independent strength components are 6 and 21 for F_i and F_{ij}, respectively.

If a material has some form of symmetry, we would expect that a number of interaction terms will vanish. For specially orthotropic materials, for example, the off-diagonal terms in Equation (5a) which are F_4, F_5 and F_6 are expected to vanish.[5] The coupling between the normal and shear strengths, e.g., F_{16}, will also vanish if we assume that the change in the sign of shear stress in Figure 1 does not change the failure stress.

By essentially the same reasoning, we can assume that shear strengths for a specially orthotropic material are all uncoupled; i.e., $F_{45}=F_{56}=F_{64}=0$. The couplings between normal strengths, however, are expected to remain. With these assumed symmetry relations, the number of independent components reduced to 3 and 9, as follows:

$$F_i = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6a}$$

$$F_{ij} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & 0 & 0 & 0 \\ & F_{22} & F_{23} & 0 & 0 & 0 \\ & & F_{33} & 0 & 0 & 0 \\ & & & F_{44} & 0 & 0 \\ & & & & F_{55} & 0 \\ & & & & & F_{66} \end{bmatrix} \tag{6b}$$

From the transformation relations in Appendix B, we can easily show that for a generally orthotropic material created by a rotation about the 3-axis, the number of nonzero components are as follows:

$$F_i' = \begin{Bmatrix} F_1' \\ F_2' \\ F_3' \\ 0 \\ 0 \\ F_6' \end{Bmatrix} \tag{7a}$$

$$F_{ij}' = \begin{bmatrix} F_{11}' & F_{12}' & F_{13}' & 0 & 0 & F_{16}' \\ & F_{22}' & F_{23}' & 0 & 0 & F_{26}' \\ & & F_{33}' & 0 & 0 & F_{36}' \\ & & & F_{44}' & F_{45}' & 0 \\ & & & & F_{55}' & 0 \\ & & & & & F_{66}' \end{bmatrix} \tag{7b}$$

Although there are 4 and 13 nonzero components in Equation (7), the number of independent components remains the same as those in Equation (6); i.e., 3 and 9 respectively.

For a transversely isotropic material with plane 2-3 as the isotropic plane, we can immediately state that indices associated with this plane are identical; i.e.,

$$F_2 = F_3, \quad F_{12} = F_{13}, \quad F_{22} = F_{33}, \quad \text{and } F_{55} = F_{66} \tag{8}$$

We can also state that the two states of stress in Figure 2 are identical and should yield identical failure stresses, then

$$F_{44} = 2(F_{22} - F_{23}) \tag{9}$$

The number of independent components reduced to 2 & 5 for F_i and F_{ij}, respectively. The number of nonzero components for this transversely isotropic material remain 3 and 9 as in Equation (6). If a rotation of the 3-axis is introduced the nonzero components of this generally transversely isotropic material also increase to 4 and 13 for F_i' and F_{ij}', respectively, as those in Equation (7).

By extending the relation of Equation (8) and (9) to the other two orthogonal planes, we obtain for isotropic materials 1 and 2 independent components for F_i and F_{ij}. If internal stresses or Bauschinger's effect is ignored, the remaining F_i component will vanish. If a failure by hydrostatic stresses is assumed to be inadmissible, the two independent components of F_{ij} become related by:

$$(F_{11} + F_{22} + F_{33}) + 2(F_{12} + F_{23} + F_{31}) = 0 \tag{10a}$$

For isotropic materials, indices 1, 2 and 3 are identical, then Equation (10a) becomes:

$$F_{12} = -F_{11}/2 \tag{10b}$$

The three shear components are also identical and by virtue of Equation (9), we obtain

$$F_{44} = F_{55} = F_{66} = 2(F_{11} - F_{12}) = 3F_{11} \tag{11}$$

Thus, for isotropic materials which are plastically incompressible and have zero internal stresses, the strength tensors are:

$$F_i \equiv \{0\} \tag{12a}$$

$$F_{ij} = F_{11} \begin{bmatrix} 1 & -1/2 & -1/2 & 0 & 0 & 0 \\ & 1 & -1/2 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 3 & 0 & 0 \\ & & & & 3 & 0 \\ & & & & & 3 \end{bmatrix} \tag{12b}$$

It is also useful to express the invariant components of the strength tensors when the axis of rotation is about the 3-axis as outlined in Appendix II:

$$F_i = \begin{Bmatrix} T_1 \\ T_2 \\ F_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{13a}$$

$$F_{ij} = \begin{bmatrix} U_1 & U_4 & V_1 & 0 & 0 & 0 \\ & U_1 & V_1 & 0 & 0 & 0 \\ & & F_{33} & 0 & 0 & 0 \\ & & & W_1 & 0 & 0 \\ & & & & W_1 & 0 \\ & & & & & 4U_5 \end{bmatrix} \tag{13b}$$

Relative strengths between two isotropic materials can be made rather directly because there is only one strength parameter associated with each isotropic material; e.g., F_{11} in Equation (12). We are not aware of any easy and direct comparison that can be made between two anisotropic materials, or between anisotropic and isotropic materials. Strength tensors are different from the elastic stiffness matrix Q_{ij} of a unidirectional composite which can be related directly to the quasi-isotropic constants of laminates.[4] For laminated composites, it may be easier to deal with a failure criterion in terms of strain components. Equation (1) can be rewritten as follows:

$$g(\varepsilon_k) = G_i\,\varepsilon_i + G_{ij}\varepsilon_i\varepsilon_j = 1 \tag{14}$$

where

$G_i = F_m C_{mi}$

$G_{ij} = F_{mn} C_{mi} C_{nj}$

C_{ij} = Elastic Stiffness Matrix

When a state of plane stress is applied to the 1-2 plane, a triclinic material will appear as follows:

$$F_i = \begin{bmatrix} F_1 \\ F_2 \\ F_6 \end{bmatrix} \qquad F_{ij} = \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ & F_{22} & F_{26} \\ & & F_{66} \end{bmatrix} \tag{15}$$

There are a total of 9 independent strength components. For specially orthotropic materials,

$$F_6 = F_{16} = F_{26} = 0 \tag{16}$$

we have 2 and 4 independent components for F_i and F_{ij}, respectively.

Engineering Strengths

The relations between engineering strengths and strength tensors are similar to those between engineering constants and the components of the elastic compliance. Typical relations for the latter are:

$$E_{11} = 1/S_{11}, \quad G_{12} = 1/S_{66} \tag{17}$$

where E_{11} is the longitudinal stiffness; G_{12}, the longitudinal shear modulus; S_{ij}, the components of the compliance matrix. An important point that has often been overlooked is that engineering constants are NOT components of a 4th rank tensor; while S_{ij} are components of such a tensor within a correction factor caused by use of the contracted notation as described in Appendix I.

Like engineering constants, engineering strengths are those strength parameters which are relatively simple to measure in the laboratory. Although they are not components of a tensor, they can be related to the components of F_i and F_{ij} through relations which we will establish and which turn out to be similar to those in Equation (17).

Let us, for example, impose a uniaxial tensile stress on a uniaxial specimen oriented along the 1-axis. We can measure the tensile failure and designate the failure stress X. Similarly, we can experimentally obtain a uniaxial compressive strength along the same axis and can designate it X'. From these two simple experiments, we obtained two engineering strengths X and X'. They can be related to the strength components F_1 and F_{11} through Equation (1) if we let i = 1 only; i.e.,

$$F_{11}\sigma_1^2 + F_1\sigma_1 = 1 \tag{18}$$

When $\sigma_1 = X$, Equation (18) becomes

$$X^2F_{11} + XF_1 = 1 \tag{19}$$

When $\sigma_1 = -X'$, Equation (18) becomes

$$X'^2F_{11} - X'F_1 = 1 \tag{20}$$

Solving Equations (19) and (20) simultaneously, we obtain

$$F_{11} = \frac{1}{XX'}$$

$$F_1 = \frac{1}{X} - \frac{1}{X'} \tag{21}$$

Through uniaxial tensile and compressive tests imposed along the 2- and 3-axis, we obtain

$$F_{22} = \frac{1}{YY'}, \quad F_2 = \frac{1}{Y} - \frac{1}{Y'} \tag{22}$$

$$F_{33} = \frac{1}{ZZ'}, \quad F_3 = \frac{1}{Z} - \frac{1}{Z'} \tag{23}$$

where Y and Y' are the uniaxial tensile and compressive strengths along the 2-axis; Z and Z', those along the 3-axis.

By imposing pure shear in the 3 orthogonal planes we can obtain

$$F_{44} = \frac{1}{QQ'}, \quad F_4 = \frac{1}{Q} - \frac{1}{Q'} \tag{24}$$

$$F_{55} = \frac{1}{RR'}, \quad F_5 = \frac{1}{R} - \frac{1}{R'} \tag{25}$$

$$F_{66} = \frac{1}{SS'}, \quad F_6 = \frac{1}{S} - \frac{1}{S'} \tag{26}$$

Where Q and Q' are positive and negative pure shear strengths along the 2-3 plane; R and R', those along the 3-1 plane; S and S', those along the 1-2 plane.

We have thus far established all 6 components of F_i and all the diagonal components of F_{ij}. The off-diagonal components are related to the interaction of two stress components in the strength criterion. For example, the experimental determinations of F_{12} and F_{16} require combined stresses. Simple uniaxial or pure shear tests will not be sufficient. Most existing strength criteria do not require combined stress tests because the interactions term such as F_{12} is assumed to be a dependent quantity or F_{16} is zero. Strength component F_{12} can be determined by an infinite number of combined stresses; only a few simple combinations will be discussed here. If we impose a biaxial tension such as

$$\sigma_1 = \sigma_2 = P, \ \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0 \tag{27a}$$

Substitute this state of combined stresses into Equation (1), we obtain

$$P^2(F_{11} + F_{22} + 2F_{12}) + P(F_1 + F_2) = 1 \tag{27b}$$

Solving for F_{12}, we obtain

$$F_{12} = \frac{1}{2P^2}\left[1 - P\left(\frac{1}{X} - \frac{1}{X'} + \frac{1}{Y} - \frac{1}{Y'}\right) - P^2\left(\frac{1}{XX'} + \frac{1}{YY'}\right)\right] \tag{27c}$$

We can determine F_{23} and F_{31} by imposing biaxial tension or compression in the 2-3 and 3-1 planes, respectively.

Strength component F_{16} can be determined by a tension-torque combination; e.g.,

$$\sigma_1 = \sigma_6 = T, \quad \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = 0 \tag{28a}$$

This is equivalent to letting i = 1 and 6 in Equation (1), and we have

$$T^2(F_{11} + F_{66} + 2F_{16}) + T(F_1 + F_6) = 1 \tag{28b}$$

by rearranging,

$$F_{16} = \frac{1}{2T^2}\left[1 - T(\frac{1}{X} - \frac{1}{X'} + \frac{1}{S} - \frac{1}{S'}) - T^2(\frac{1}{XX'} + \frac{1}{SS'})\right] \tag{28c}$$

This experiment can be readily performed by testing a tubular specimen with the tube axis along the 1-axis. Component F_{26} can be determined by imposing tension-torque on a tubular specimen with the 1-axis along the circumference of the tube. Similar to the case of biaxial tension where the ratio of the two normal stresses may be arbitrary, the tension-torque combination can also have ratios different from that of Equation (28a), where the ratio was unity. These different ratios will provide additional determinations of the interaction terms. The redundant measurements can be used to establish the range of validity and accuracy of our initial assumption stated in Equation (1). If the tube axis coincide with the material symmetry axis, e.g., the filament

axis of a unidirectional composite runs along the longitudinal direction of the tube, F_{16} in Equation (28c) must be zero in this specially orthotropic orientation. Then we can easily establish the relation between the imposed combined stresses T and the principal strength components X, X' and S. S' is equal to S for specially orthotropic material based on the relations in Equations (7a) and (26).

Many Russian workers recommend the use of 45-degree specimens for the determination of interaction terms such as F_{12}. This can be done by letting:

$$\sigma_1 = \sigma_2 = \sigma_6 = U/2, \quad \sigma_3 = \sigma_4 = \sigma_5 = 0 \tag{29a}$$

where U is the tensile strength of a 45-degree off-axis specimen. Note that the combined stresses in Equation (29a) are applied to the symmetry axes of a specially orthotropic material. This state of stress is equivalent to a uniaxial tensile stress applied to a reference coordinate system rotated 45 degrees from the material symmetry axes. This is why U can be considered as an engineering strength. Care must be exercised in the actual experiment of loading a 45-degree specimen so that the shear coupling effect due to S_{16} is minimized. By introducing Equation (29a) into Equation (1), we have

$$\frac{U^2}{4}(F_{11} + F_{22} + F_{66} + 2F_{12}) + \frac{U}{2}(F_1 + F_2) = 1 \tag{29b}$$

where F_{16}, F_{26}; and F_6 vanish because of the assumed special orthotropy. Now we can obtain F_{12} by this test:

$$F_{12} = \frac{2}{U^2}\,[1 - \frac{U}{2}\,(\frac{1}{X} - \frac{1}{X'} + \frac{1}{Y} - \frac{1}{Y'})$$

$$- \frac{U^2}{4}(\frac{1}{XX'} + \frac{1}{YY'} + \frac{1}{SS'})] \tag{29c}$$

A similar relation for the compressive strength U' can be established. A similar relation for the compressive strength U' of a 45 degree off-axis unidirectional specimen is as follows:

$$\sigma_1 = \sigma_2 = \sigma_6 = -U'/2, \quad \sigma_3 = \sigma_4 = \sigma_5 = 0 \tag{30a}$$

$$\frac{U'^2}{4}\,(F_{11} + F_{22} + 2F_{12} + F_{66}) - \frac{U'}{2}(F_1 + F_2) = 1 \tag{30b}$$

Then we can find

$$F_{12} = \frac{2}{U'^2}\,[1 + \frac{U'}{2}\,(\frac{1}{X} - \frac{1}{X'} + \frac{1}{Y} - \frac{1}{Y'})$$

$$- \frac{U'^2}{4}\,(\frac{1}{XX'} + \frac{1}{YY'} + \frac{1}{S^2}\,)] \tag{30c}$$

By comparing Equation (29b) and (30b), we can derive the following relations between U and U':

$$2(\frac{1}{U} - \frac{1}{U'}) = F_1 + F_2 = 2T_1 \tag{31a}$$

$$\frac{4}{UU'} = F_{11} + F_{22} + 2F_{12} + F_{66} = 4(U_1 - U_3) \tag{31b}$$

where T_1 is the first invariant of F_i which is shown in Appendix II, Equation (91) and Figure 3 of this paper; while the difference on the left-hand side Equation (31b) is not invariant which can be seen from Equation (90).

Let V and V' be the positive and negative shear strengths of a 45 degree off-axis unidirectional specimen, then analogous to the relations in Equations (29) through (31) for uniaxial stresses, we have

$$+\sigma_1 = -\sigma_2 = V, \quad \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0 \tag{32a}$$

This state of stress is applied to the symmetry axes oriented at +45 degrees from the positive shear stress V. The same state of stress exists when a negative shear stress (-V) is applied to a -45 degree off-axis specimen. Substituting Equation (32a) into (1), we have

$$V^2(F_{11} + F_{22} - 2F_{12}) + V(F_1 - F_2) = 1 \tag{32b}$$

Then we can establish

$$F_{12} = -\frac{1}{2V^2}\left[1 - V\left(\frac{1}{X} - \frac{1}{X'} - \frac{1}{Y} + \frac{1}{Y'}\right) - V^2\left(\frac{1}{XX'} + \frac{1}{YY'}\right)\right] \tag{32c}$$

Similarly, when

$$+\sigma_1 = -\sigma_2 = -V', \quad \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0 \tag{33a}$$

We have

$$V'^2 (F_{11} + F_{22} - 2F_{12}) - V'(F_1 - F_2) = 1 \tag{33b}$$

$$F_{12} = -\frac{1}{2V^2}\left[1 + V'(\frac{1}{X} - \frac{1}{X'} - \frac{1}{Y} + \frac{1}{Y'}) - V'^2(\frac{1}{XX'} + \frac{1}{YY'})\right] \tag{33c}$$

From Equations (32b) and (33b) we can derive the following relations between V and V':

$$\frac{1}{V} - \frac{1}{V'} = F_1 - F_2 = 2T_2 \tag{34a}$$

$$\frac{1}{VV'} = F_{11} + F_{22} - 2F_{12} \tag{34b}$$

All these relations are helpful in determining components of F_{ij} and their transformed quantities as well as the internal consistency of this present theory. From Equations (31b) and (34b), we can derive an invariant relation:

$$\frac{1}{VV'} - \frac{4}{UU'} = 4(U_3 - U_4) \tag{35}$$

We will describe later in this paper how component F_{12} for a given composite system, e.g., the graphite-epoxy composite, can be best determined. Suffice to say, F_{12} is a very sensitive and critical quantity in this proposed theory and must be clearly understood by its users.

Quadratic Approximations

Using the framework and notation of our approach, we can compare the forms of many existing quadratic approximations of the strength criteria. The Hill criterion[3] which is limited to specially orthotropic 3-dimensional bodies, with plastic incompressibility and without internal stresses, can be expressed in the following forms:

$$F_i \equiv \{0\} \tag{36}$$

$$F_{ij} = \begin{bmatrix} \frac{1}{X^2} & -\frac{1}{2}\left(\frac{1}{X^2}+\frac{1}{Y^2}-\frac{1}{Z^2}\right) & -\frac{1}{2}\left(\frac{1}{Z^2}+\frac{1}{X^2}-\frac{1}{Y^2}\right) & 0 & 0 & 0 \\ & \frac{1}{Y^2} & -\frac{1}{2}\left(\frac{1}{Y^2}+\frac{1}{Z^2}-\frac{1}{X^2}\right) & 0 & 0 & 0 \\ & & \frac{1}{Z^2} & 0 & 0 & 0 \\ & & & \frac{1}{Q^2} & 0 & 0 \\ & & & & \frac{1}{R^2} & 0 \\ & & & & & \frac{1}{S^2} \end{bmatrix} \tag{37}$$

Since the Hill criterion is obtained by generalizing the Mises criterion, 3 interaction terms become dependent on the diagonal terms. Such arbitrary generalization resulted in only 6 independent strength components in F_{ij} instead of 9. The Mises criterion can be recovered from Equation (37) if we let

$$X^2 = Y^2 = Z^2 = Q^2/3 = R^2/3 = S^2/3 \tag{38}$$

Then Equation (37) becomes the same as Equation (12) with F_{11} replaced by $1/X^2$. Substituting the latter result in Equation (1), we have

$$\begin{aligned}\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 \\ + 3(\sigma_4^2 + \sigma_5^2 + \sigma_6^2) = X^2\end{aligned} \tag{39}$$

or

$$\begin{aligned}(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \\ + 6(\sigma_4^2 + \sigma_5^2 + \sigma_6^2) = 2X^2 = 6S^2\end{aligned} \tag{40}$$

Note that plastic incompressibility is satisfied in Equation (40).

Some authors have tried to generalize the quadratic approximation derived from Equation (37) by introducing floating or adjustible constants for the off-diagonal or interaction terms such as[7]

$$F_{12} = a(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}) \tag{41}$$

The use of constant "a" implies that F_{12} is proportional to a particular function of the engineering strengths which is in general not the case. As stated earlier, the generalization of the Mises criterion to describe special orthotropy lacks analytic foundation. Further generalization from the form of Equation (37) by means of adjustible constants of proportionality may lead to unnecessarily restrictive, if not erroneous, strength criteria. The use of arbitrary constants, like "a" in Equation (41), does not ensure internal consistency and invariance under transformation.

In fact, when the strength criterion based on Equation (37) is specialized to plane stress applied to a unidirectional composite with the longitudinal strength X in the 1-direction, the two transverse strengths are equal; i.e.,

$$Y = Z = \text{Transverse Strength} \tag{42}$$

With this additional assumption, we can derive the often-used strength criterion

$$\frac{\sigma_1^2}{X^2} - \frac{\sigma_1\sigma_2}{X^2} + \frac{\sigma_2^2}{Y^2} + \frac{\sigma_6^2}{S^2} = 1 \tag{43}$$

Note that this criterion is biased; i.e., the interaction term contains only the longitudinal strength. If we are not careful and apply the plane-stress strength criterion to the same unidirectional composite but with filaments running in the 2-direction, we will obtain the following criterion:

$$\frac{\sigma_1^2}{Y^2} - \frac{\sigma_1\sigma_2}{Y^2} + \frac{\sigma_2^2}{X^2} + \frac{\sigma_6^2}{S^2} = 1 \tag{44}$$

This latter equation is not a correct specialization of the strength tensor in Equation (37) because the equivalent relation of Equation (42) no longer holds; i.e.,

$$X \neq Z \tag{45}$$

For the same reason, the plane-stress strength criterion cannot be applied to a laminate because of the strength transverse to the plane of the laminate is an independent engineering strength, entirely unrelated to the in-plane engineering strength; i.e., $\overline{X}$, $\overline{Y}$, and $\overline{Z}$ are all independent where the bar designates the equivalent engineering strength of a laminate.

With the same number of assumptions as Hill's, i.e., plane stress, special orthotropy, and no internal stresses, we can correct the deficiencies of Equation (43) by using the proper interaction term F_{12} from Equation (27c) or (29c) or other combined stress engineering strengths. Instead of $1/X^2$ for the interaction term, we have,

$$F_{12} = \frac{1}{2}\left(\frac{1}{P^2} - \frac{1}{X^2} - \frac{1}{Y^2}\right), \tag{46}$$

or

$$F_{12} = 2\frac{1}{U^2} - \frac{1}{4}\left(\frac{1}{X^2} + \frac{1}{Y^2} + \frac{1}{S^2}\right) \tag{47}$$

These relations are derived from Equations (27c) and (29c), respectively. An additional engineering strength, P or U, is

needed to complete the strength criterion. By making one more assumption, such as P = Y in Equation (46), we can obtain the often-used relation of Equation (43). The same improvement can be made to the strength criterion of Hoffman[8]; i.e., change the interaction term $1/X^2$ to that in Equation (27c) or (29c).

Transformation of Strength Tensors

In order to gain insight into our strength criterion, it is helpful if we examine the transformed properties of the strength tensors. We will show graphically the transformed properties of unidirectional graphite-epoxy composites with the following engineering strengths:

$$\begin{aligned}
X &= \text{Longitudinal Tension} = 150 \text{ ksi} \\
X' &= \text{Longitudinal Compression} = 100 \text{ ksi} \\
Y &= \text{Transverse Tension} = 6 \text{ ksi} \\
Y' &= \text{Transverse Compression} = 17 \text{ ksi} \\
S &= \text{Longitudinal Shear} = 10 \text{ ksi} \\
F_{12} &= \pm\sqrt{F_{11}F_{22}} = \pm.0008084
\end{aligned} \tag{48}$$

Assuming that the graphite unidirectional composite is specially orthotropic and under plane stress, the principal strength components are:

$$F_i = \begin{Bmatrix} F_1 \\ F_2 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{X} - \frac{1}{X'} \\ \frac{1}{Y} - \frac{1}{Y'} \\ 0 \end{Bmatrix} \tag{49}$$

$$F_{ij} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ \cdot & F_{22} & 0 \\ & & F_{66} \end{bmatrix} = \begin{bmatrix} \frac{1}{XX'} & F_{12} & 0 \\ & \frac{1}{YY'} & 0 \\ \cdot & & \frac{1}{S^2} \end{bmatrix} \tag{50}$$

where F_{12} is imposed by the stability condition of Eq. (3).

The transformation of the strength tensor from Equation (91) are:

$$\begin{aligned} T_1 &= (F_1 + F_2)/2 = .0523 \\ T_2 &= (F_1 - F_2)/2 = -.0556 \\ T_3 &= F_6 = 0 \end{aligned} \tag{51}$$

Thus,

$$\begin{aligned} F_1' &= .0523 - .0556 \cos 2\theta \\ F_2' &= .0523 + .0556 \cos 2\theta \\ F_6' &= -.1112 \sin 2\theta \end{aligned} \tag{52}$$

The graphical representation of Equation (52) is shown in Figure 3. Similarly the transformation of F_{ij} can be obtained directly from the relations in Equation (92) that

$$\begin{aligned} U_1 &= \tfrac{1}{8}(3F_{11} + 3F_{22} + 2F_{12} + F_{66}) = .00515 \\ U_2 &= \tfrac{1}{2}(F_{11} - F_{22}) = -.00487 \\ U_3 &= \tfrac{1}{8}(F_{11} + F_{22} - 2F_{12} - F_{66}) = -.00022 \end{aligned} \tag{53}$$

$$U_4 = \frac{1}{8}(F_{11} + F_{22} + 6F_{12} - F_{66}) = .00059$$
$$U_5 = \frac{1}{8}(F_{11} + F_{22} - 2F_{12} + F_{66}) = .00228 \tag{53}$$

The transformed components are:

$$F_{11}' = (5.15 - 4.87 \cos 2\theta - 0.22 \cos 4\theta) \times 10^{-3}$$
$$F_{12}' = (0.59 + 0.22 \cos 4\theta) \times 10^{-3}$$
$$F_{66}' = (9.12 + 0.87 \cos 4\theta) \times 10^{-3} \tag{54}$$
$$F_{16}' = (0.49 \sin 2\theta + 0.44 \sin 4\theta) \times 10^{-3}$$

Components F_{22}' and F_{26}' can be obtained from F_{11}' and F_{16}', respectively, by changing θ in Equation (54) to $-(\theta+90)$. Graphic representations of F_{ij}' are shown in Figure 4.

The transformed properties shown in Figures 3 and 4 are typical of 2nd and 4th rank tensors. The invariants associated with this particular transformation, i.e., a rotation about the 3-axis, are also shown as horizontal lines.

The solid lines represent the upper bound of F_{12}, i.e., F_{12} = +.0008, and the dashed lines, the lower bound, i.e., F_{12} = -.0008. The present theory can only admit limited values of F_{12} for the stability reason depicted by Equation (3).

Similar constraint must be imposed on all the remaining interactions on off-diagonal terms of F_{ij}. If the magnitude of F_{12}, for example, exceeds the limits indicated by Equation (3), the resulting yield surface may become hyperboloidal. The off-axis properties in case of a F_{12} outside the stable range may

have several maxima or minima which do not look like the expected well-behaved functions. In other cases, the off-axis properties "blow up" at certain angles. This was shown by Ashkenazi[9] when he substituted an unconstrainted value for F_{12} into the theory proposed by Gol'denblat and Kopnov[2]. Thus the initial assumption of Equation (1) in this paper and that employed in Reference 2 must be further constrained by some stability considerations. If experimental data do not agree with the predictions and constraints of our theory, we can modify the initial assumptions, such as the change in the functional form of Equation (1) or the inclusion of high order terms, but we are not at liberty to relax the stability requirement.

The determination of the value of F_{12} can be achieved through infinite number of combined-stress states, shown earlier in this paper. The Russian workers[2,9] suggested the use of off-axis tests, with the 45-degree specimen as the most popular choice. It is interesting to observe the effect of these 45-degree uniaxial (U,U'), pure shear (V,V') and hydrostatic (P,P') tests on the value of F_{12}. In Figure 5, we used the first five data for graphite-epoxy composites listed in Equation (49) and various values of F_{12}. These values are substituted into Equations (27) and (29) through (33) to obtain the curves in Figure 5 for various off-axis properties U, U', V, V', P and P'. Curves such as U, P and V' are nearly horizontal which means that those tests should not be used for the determination of F_{12}. A small inaccuracy in the values of U, for example, can induce a large change in F_{12}. Of the remaining curves, the V (positive shear

of a 45-degree specimen) gives the most reliable determination of F_{12}.

Also shown in Figure 5 are the limits imposed by the stability conditions of Equation (3). For the remaining part of this paper, we will use a F_{12} value close to its upper bound, say, +.0008, since the positive shear strength for our experimental measurement of 45-degree specimen is about 20 ksi. It should be emphasized that Figure 5 is valid for a particular composite with engineering strengths shown in Equation (49). For other composites, we believe that similar study on the sensitivity of off-axis or other combined-stress tests on F_{12} should be made before a reasonable value of F_{12} is decided upon.

Off-Axis Properties

If we want to obtain the off-axis uniaxial strengths, we simply use the transformed strength tensors in Equation (18); i.e.,

$$F_{11}'\sigma_1^2 + F_1'\sigma_1 - 1 = 0 \tag{55}$$

The transformed strength tensors are shown in Figures 3 and 4. At 45 degrees, for F_{12} = +.0008 for example,

$$\begin{aligned} F_{11}' &= .00537 \\ F_1' &= .05225 \end{aligned} \tag{56}$$

Solving the quadratic equation, we obtain two roots:

$$\sigma_1 = +9.61 \text{ and } -19.35 \text{ ksi} \tag{57}$$

If the process is repeated for other angles between zero and 90 degrees, we obtain the off-axis uniaxial properties shown in Figure 6a. Like the transformed Young's modulus E_{11}', and off-axis uniaxial properties, they do not follow the transformation of any known tensors. They are therefore by definition not tensors.

As another comparison, we also show in dashed lines in Figure 6b the prediction of the maximum stress theory which can be expressed analytically by one of the following 6 criteria whichever happens to be minimum for a given state of plane stress:

For uniaxial tensions:

$$\sigma_1 = X/m^2,\ Y/n^2, \text{ or } S/mn \tag{58}$$

For uniaxial compression:

$$\sigma_1 = X'/m^2,\ Y'/n^2, \text{ or } S'/mn \tag{59}$$

where $m = \cos\theta$, $n = \sin\theta$. The values of the engineering strengths are the same as those shown in Equation (49), except that only 5 of the 6 strengths are needed for the maximum stress theory. The biaxial tension P, for example, is assumed to be derivable from the 6 strength criteria shown in Equations (58) and (59).

The off-axis pure shear properties can also be obtained by solving the following quadratic equation for various values of angle θ;

$$F_{66}' \sigma_6^2 + F_6' \sigma_6 - 1 = 0 \tag{60}$$

Where the transformed strength tensors can be obtained from Figures 3 and 4. The results for the graphite composite with the same principal strengths in Equation (49) are shown in Figure 7a The corresponding predictions based on the maximum stress theory are shown as dashed lines in Figure 7b and they are derived from the following relations:

For positive shear:

$$\sigma_6 = X/2mn, \quad Y'/2mn, \text{ or } S/(m^2-n^2) \tag{61}$$

For negative shear:

$$\sigma_6 = X'/2mn, \quad Y/2mn, \text{ or } S'/(m^2-n^2) \tag{62}$$

Although we have not shown the strength prediction based on the maximum strain theory in Figures 6 and 7, it is very similar in nature to the prediction of the maximum stress theory. There will be 6 simultaneous criteria for the case of plane stress. The lowest predicted strength among the 6 relations will govern for each state of combined stresses. The final criterion will consist of segmented curves as those shown in Figures 6 and 7. Each stress or strain criterion is unaffected by the presence or absence of other stress or strain components; i.e., there is no interaction among the stress or strain components.

From the shapes of the off-axis uniaxial and shear curves in Figures 6 and 7, we should be able to fit available test data very closely. Data points are shown in the Figures. Without the analytic foundation contained in the initial postulate in Equation (1), it is very difficult, if not impossible, to deduce the transformation properties associated with the strength of a composite from experimental data like those shown in Figures 6 and 7. The strength criterion proposed here certainly contains greater flexibility than most existing criteria while it maintains the necessary generality in spatial dimensions, materials symmetries, combined-stress state invariance, and, above all, operational simplicity.

Transverse Shears

As applications of composites for primary structures increase, the cross-sectional thickness is also increasing. The effects of transverse shears σ_4 and σ_5 in a thick laminate have been the subject of several recent articles. If we confine ourselves to unidirectional layers oriented at various angles through a rotation about the 3-axis from the specially transversely isotropic orientation, our strength tensors are those in Equation (6), (8) and (9) which in terms of engineering strengths are as follows:

$$F_i = \begin{Bmatrix} \frac{1}{X} - \frac{1}{X'} \\ \frac{1}{Y} - \frac{1}{Y'} \\ \frac{1}{Y} - \frac{1}{Y'} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{63a}$$

$$F_{ij} = \begin{bmatrix} \frac{1}{XX'} & F_{12} & F_{12} & 0 & 0 & 0 \\ & \frac{1}{YY'} & F_{23} & 0 & 0 & 0 \\ & & \frac{1}{YY'} & 0 & 0 & 0 \\ & & & 2(\frac{1}{YY'} - F_{23}) & & 0 \\ & & & & \frac{1}{S^2} & 0 \\ & & & & & \frac{1}{S^2} \end{bmatrix} \tag{63b}$$

We have assumed in the above that the 2-3 plane is the isotropic plane. Only the interaction terms F_{12} and F_{23} are unfamiliar quantities. There is however no intrinsic difficulty associated with their experimental determination. From the principal components of Equation (63) we can readily find the components of the strength tensors for any orientations about the 3-axis. The off-symmetry tensors will have more nonzero components than the above. The exact number of the nonzero components are shown in Equation (7). Components F_{12} and F_{13} are equal in Equation (63) which refers to the symmetry axes. But for off-symmetry orientations, F_{12}' and F_{13}' are in general different. Their transformation relations are also different as can be seen in Appendix II. The same can be said about the isotropic relations among the components associated with the 2-3 plane; viz., F_{22}, F_{23}, F_{33}, and F_{44}. The relations shown in Equation (63) are valid only in the symmetry axes.

Application of our strength criterion to a 3-dimensional transversely isotropic material can be achieved in two ways. First, we can transform the stress components to the material-symmetry axes, in which case Equation (1) can be expanded containing only the independent and nonzero components shown in Equations (63), we have

$$F_1\sigma_1' + F_2(\sigma_2' + \sigma_3') + F_{11}\sigma_1'^2 + F_{22}(\sigma_2'^2 + \sigma_3'^2 + 2\sigma_4'^2)$$

$$+ \quad F_{66}(\sigma_5'^2 + \sigma_5'^2) + 2F_{12}(\sigma_1'\sigma_2' + \sigma_1'\sigma_3') + 2F_{23}(\sigma_2'\sigma_3' - 2\sigma_4'^2)$$

$$= \quad 1 \qquad (64)$$

Conceptually, this strength criterion is no more difficult than that for the plane stress. The most difficult job is the analysis that determines the additional stress components σ_3, σ_4, and σ_5. Once they are known the strength criterion, such as Equation (64), can be applied directly.

The other method of applying the strength criterion is to leave the stress components unchanged but rotate the material symmetry axes. We must then find out the transformed strength tensors, which will appear like Equation (7) in their nonzero components. Since we will have more strength components in the generally transversely isotropic configuration than the 7 independent components in Equation (64), Equation (1) when expanded for this case will have more terms. There will be the following 17 terms to be exact:

$$\begin{aligned} &F_1'\sigma_1 + F_2'\sigma_2 + F_3'\sigma_3 + F_6'\sigma_6 + F_{11}'\sigma_1^2 + F_{22}'\sigma_2^2 + F_{33}'\sigma_3^2 \\ &+ F_{44}'\sigma_4^2 + F_{55}'\sigma_5^2 + F_{66}'\sigma_6^2 + 2F_{12}'\sigma_1\sigma_2 + 2F_{23}'\sigma_2\sigma_3 \\ &+ 2F_{31}'\sigma_3\sigma_1 + 2F_{16}'\sigma_1\sigma_6 + 2F_{26}'\sigma_2\sigma_6 + 2F_{36}'\sigma_3\sigma_6 \\ &+ 2F_{45}'\sigma_4\sigma_5 = 1 \end{aligned} \tag{65}$$

Again, this strength criterion is conceptually simple. Components F_i' and F_{ij}' are obtained by transforming the principal components in Equation (63).

Conclusions

An operationally simple strength criterion can be developed using a scalar function of two strength tensors. This criterion is an improvement over most existing quadratic approximations of the yield surface because of the use of strength tensors. We can extend our working experience of elastic moduli and compliance to the strength criterion. The transformation relations and the associated invariants are well known. The number of independent and nonzero strength components for each material symmetry are also simple extensions of those governing the elastic properties of anisotropic materials. Spatial dimensions of the body and the states of combined stresses can be treated in a unified manner. Although available experimental data in combined-stress states are only tentative, our strength criterion should fit them better than most, if not all, existing curve-fitting schemes. We are in the process of generating more data and it is our hope that the additional data will demonstrate more conclusively the utility of our criterion than we are able to do at this time.

Finally, we have tried to show the interaction term F_{12} as an independent but constrained strength component. It is an essential variable that makes the strength tensor possible. For the graphite-epoxy composite presented in this paper, the quadratic approximation of the strength criterion proposed by Hoffman[8], which is an improvement over Hill's[3] by incorporating the

effect of internal stresses, has an interaction term so small (-.00007) that it can be treated as zero. This can be seen in Figure 5. For highly anisotropic composites, the off-axis tensile test does not appear to be an effective way to compare the difference between various strength theories, which include the maximum stress, Hoffman and ours. Off-axis compression data would be better. A positive shear such as V in Figure 5 is the best.

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APPENDIX I

CONTRACTED NOTATION

Contracted notation is an artificial system which offers the advantages of

1. one half of the number of indices, i.e., one index for second-rank; and two, for 4th-rank tensors; and
2. retaining the range and summation conventions of the indicial notation.

But in the process of simplifications via the contracted notation, several difficulties arise,

1. there is no universal agreement as to the order of the off-diagonal terms, e.g., the shear stress σ_{12} can be represented by σ_4 or σ_6;
2. Multiplying factors are often required, such as

$$\begin{aligned} S_{66} &= 4S_{1212} \\ F_6 &= 2F_{12} \end{aligned} \tag{66}$$

where S_{ij} is the elastic compliance matrix and F_i is the strength tensor in Equation (1). When various factors are applied to tensor components such as those on the right-hand side of Equation (66) the components of a contracted tensors like S_{66} and F_6 are no longer tensors in the same way that engineering strain differs from tensorial strain by a factor of 2 in the shear components.

It is the purpose of this Appendix to show the origins of the multiplying factors of a few common tensors including the strength tensors of this paper. The only way to ensure reliability in the use of contracted notation is to do the long-hand

or natural notation and compare it with the result of the contracted notation. The same practice is common in the use of indicial notation; i.e., compare indicial notation against the long-hand operations in terms of a coordinate system.

Second Rank Tensors

We would like to adopt the ordering and conversion scheme for stress and strain used by Love, Sokolniskoff, Green, Hearmon;

Natural Notation		Contracted Notation	
σ_{11}	ε_{11}	σ_1	ε_1
σ_{22}	ε_{22}	σ_2	ε_2
σ_{33}	ε_{33}	σ_3	ε_3
σ_{23}	$2\varepsilon_{23}$	σ_4	ε_4
σ_{31}	$2\varepsilon_{31}$	σ_5	ε_5
σ_{12}	$2\varepsilon_{12}$	σ_6	ε_6
σ_{32}	$2\varepsilon_{32}$	σ_7	ε_7
σ_{13}	$2\varepsilon_{13}$	σ_8	ε_8
σ_{21}	$2\varepsilon_{21}$	σ_9	ε_9

(67)

The conversion system above is not followed by all authors. The conversion of stress maintains the tensorial property; but that of strain does not. The factor of 2 on the shear strain

components makes the strain in contracted notation engineering rather than tensorial strain. Thermal expansion coefficients α_i will have to make the same correction as strain components in Equation (66), if the thermal and mechanical strains are additive in thermal stress analysis such as

$$\varepsilon_i^{total} = \varepsilon_i^m + \alpha_i T \tag{68}$$

Similarly, in laminated plate analysis, it is common to assume

$$\varepsilon_i = \varepsilon_i^o + zk_i \tag{69}$$

Where ε_i^o is misplaced in-plane strain and k_i, curvature, then k_i must also be in engineering curvature, i.e.,

$$k_6 = 2k_{12} = - \frac{2\partial^2 w}{\partial w \partial y} \tag{70}$$

In a scalar multiplication such as strain energy in-plane stress and natural notation, we have with i,j = 1, 2

$$W = \frac{1}{2} (\sigma_{ij}\varepsilon_{ij}) = \frac{1}{2} (\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + 2\sigma_{12}\varepsilon_{12}) \tag{71}$$

In contracted notation with i = 1, 2, 6

$$W = \frac{1}{2} \sigma_i \varepsilon_i = \frac{1}{2} (\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_6\varepsilon_6) \tag{72}$$

The last 2 equations are equal if and only if the use of contracted notation follows that shown in Equation (66), i.e., the key relation of

$$2\varepsilon_{12} = \varepsilon_6 \tag{73}$$

The first scalar product in Equation (1), $F_i\sigma_i$, in natural notation with i,j = 1, 2, 3 is:

$$F_{ij}\sigma_{ij} = F_{11}\sigma_{11} + F_{22}\sigma_{22} + F_{33}\sigma_{33} + (F_{23} + F_{32})\sigma_{23} + (F_{31} + F_{13})\sigma_{13} + (F_{12} + F_{21})\sigma_{12} \tag{74}$$

Where the stress is symmetrical. The same scalar product in contracted notation with i = 1,...6 is

$$F_i\sigma_i = F_1\sigma_1 + F_2\sigma_2 + F_3\sigma_3 + F_4\sigma_4 + F_5\sigma_5 + F_6\sigma_6 \tag{75}$$

The last two equations are equal if the equivalence between the natural and contracted notation follows that for the strains in Equation (66). Since stress is symmetrical, whether or not F_{ij} in Equation (74) is symmetrical is no longer distinguishable in Equation (75); i.e.,

$$F_6 = F_{12} + F_{21} \tag{76}$$

From Equation (2) we know $F_{ijk\ell}$ is symmetric, f is class C2, we can conclude in natural notations

$$F_{ij} = \frac{\partial f}{\partial\sigma_{ij}} - F_{ijk\ell}\sigma_{k\ell} = F_{ji} \tag{77}$$

Thus in contracted notation,

$$F_4 = F_7 = 2F_{23} \quad F_5 = F_8 = 2F_{31} \quad F_6 = F_9 = 2F_{12} \tag{78}$$

Wince we also know from Equations (2) that F_{ij} is symmetric both strength tensors are symmetric. We can also conclude that

the transformations of α_i, k_i and F_i will all follow that of the engineering strain. The exact relationship for F_i will appear in Appendix II.

Fourth Rank Tensors

Using the contracted notation of Equation (66,67), for stress and strain, we can determine the multiplying factors of the stiffness and compliance matrix, C_{ij} and S_{ij}, respectively. In their natural forms:

$$\sigma_{ij} = C_{ijk\ell}\varepsilon_{k\ell} \qquad \varepsilon_{ij} = S_{ijk\ell}\sigma_{k\ell} \tag{79}$$

Where i,j = 1,2,3. In matrix form, we have

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{21} \end{Bmatrix} = [C_{ijk\ell}] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{32} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix} \tag{80}$$

If both stress and strain are symmetric, we have

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \left[\begin{array}{c:c} C_{ijk\ell} & 2C_{ijk\ell} \\ \hdashline C_{ijk\ell} & 2C_{ijk\ell} \end{array}\right] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{Bmatrix} = [C_{ijk\ell}] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{Bmatrix} \tag{81}$$

If contracted notation in Equation (A-2) is introduced, we have

$$\begin{Bmatrix} \sigma_{1} \\ \sigma_{2} \\ \sigma_{3} \\ \sigma_{4} \\ \sigma_{5} \\ \sigma_{6} \end{Bmatrix} = [C_{ijk\ell}] \begin{Bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{3} \\ \varepsilon_{4} \\ \varepsilon_{5} \\ \varepsilon_{6} \end{Bmatrix} \tag{82}$$

Note that the factor of 2 in the last 3 columns of $C_{ijk\ell}$ in Equation (81) due to the symmetry of strain is cancelled by the engineering strain used in the contracted notation.

We can write the equivalent of (80) for $S_{ijk\ell}$. When stress and strain are symmetric, we have in place of (81)

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{Bmatrix} = \left[\begin{array}{c|c} S_{ijk\ell} & 2S_{ijk\ell} \\ \hline S_{ijk\ell} & 2S_{ijk\ell} \end{array}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} \tag{83}$$

If we now introduce contracted notation of Equation (67), we have

$$\begin{Bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{3} \\ \varepsilon_{4} \\ \varepsilon_{5} \\ \varepsilon_{6} \end{Bmatrix} = \left[\begin{array}{c|c} S_{ijk\ell} & 2S_{ijk\ell} \\ \hline 2S_{ijk\ell} & 4S_{ijk\ell} \end{array}\right] \begin{Bmatrix} \sigma_{1} \\ \sigma_{2} \\ \sigma_{3} \\ \sigma_{4} \\ \sigma_{5} \\ \sigma_{6} \end{Bmatrix} \tag{84}$$

Note the multiplying factors of 1, 2 and 4 applied to the four equal 3 x 3 submatrices of $S_{ijk\ell}$. These factors are contributed equally from two sources; viz., symmetrical stress <u>and</u> engineering strain.

The scalar product of the quadratic term in Equation (1) should be the same as the strain energy in Equation (71)

$$W = \frac{1}{2} (\sigma_{ij}\varepsilon_{ij}) = \frac{1}{2} S_{ijk\ell}\sigma_{ij}\sigma_{k\ell} \tag{85}$$

or

$$f = F_{ijk\ell}\sigma_{ij}\sigma_{k\ell} \tag{86}$$

In contracted notation, we have

$$W = \frac{1}{2}\sigma_i\varepsilon_i = \frac{1}{2}S_{ij}\sigma_i\sigma_j \tag{87}$$

or

$$f = F_{ij}\sigma_i\sigma_j \tag{88}$$

If we expand (85) and (87) and match each term, we will get the same multiplying factors as those shown in Equation (84). The same multiplying factors will apply to $F_{ijk\ell}$ as follows:

Natural Notation	Contracted Notation
F_{1111}, F_{2222}, F_{3333}	F_{11}, F_{22}, F_{33}
F_{2233}, F_{3311}, F_{1122}	F_{23}, F_{31}, F_{12}
$2F_{1123}$, $2F_{1131}$, $2F_{1112}$	F_{14}, F_{15}, F_{16}
$2F_{2223}$, $2F_{2231}$, $2F_{2212}$	F_{24}, F_{25}, F_{26}
$2F_{3323}$, $2F_{3331}$, $2F_{3312}$	F_{34}, F_{35}, F_{36}
$4F_{2323}$, $4F_{3131}$, $4F_{1212}$	F_{44}, F_{55}, F_{66}
$4F_{2331}$, $4F_{3112}$, $4F_{1223}$	F_{45}, F_{56}, F_{64}

(89)

Thus, F_{ij} in contracted form will transform like S_{ij} with proper multiplying factors in Equation (89) or (84). The exact relations will be listed in Appendix II.

APPENDIX II

TRANSFORMATION RELATIONS OF STRENGTH TENSORS

Transformation relations for tensors can be expressed in terms of multiple angles if the following trignometric identities are used:

	Constant	$\cos 2\theta$	$\sin 2\theta$	$\cos 4\theta$	$\sin 4\theta$	
m^2	1/2	1/2	0	0	0	
n^2	1/2	-1/2	0	0	0	
m^2-n^2	0	1	0	0	0	
mn	0	0	1/2	0	0	
m^4	3/8	1/2	0	1/8	0	
m^3n	0	0	1/4	0	1/8	
m^2n^2	1/8	0	0	-1/8	0	
mn^3	0	0	1/4	0	-1/8	(90)
n^4	3/8	-1/2	0	1/8	0	
m^4+n^4	3/4	0	0	1/4	0	
m^4-n^4	0	1	0	0	0	
m^3n-mn^3	0	0	0	0	1/4	
m^3n+mn^3	0	0	1/2	0	0	
$m^4-3m^2n^2$	0	1/2	0	1/2	0	
$3m^2n^2-n^4$	0	1/2	0	-1/2	0	
$(m^2-n^2)^2$	1/2	0	0	1/2	0	

When the relations above are substituted into the classical transformation relations of a rotation about the 3-axis, as shown in Figure 8 we have the following relations. Since

all components are expressed in terms of contract notation, the multiplying factors described in Appendix II have been incorporated.

I. Second Rank Tensor F_i

	Constant	$\cos\theta$	$\sin\theta$	$\cos 2\theta$	$\sin 2\theta$
F'	T_1	0	0	T_2	$\frac{1}{2} F_6$
F_2'	T_1	0	0	$-T_2$	$-\frac{1}{2} F_6$
F_6'	0	0	0	F_6	$-2T_2$
F_4'	0	F_4	F_5	0	0
F_5'	0	F_4	F_5	0	0
F_3'	F_3	0	0	0	0

(91)

where $T_1 = (F_1 + F_2)/2$

$T_2 = (F_1 - F_2)/2$

II. Fourth Rank Tensor F_{ij}

	Constant	cos 2θ	sin 2θ	cos 4θ	sin 4θ
F_{11}'	U_1	U_2	$2U_6$	U_3	U_7
F_{22}'	U_1	$-U_2$	$-2U_6$	U_3	U_7
F_{12}'	U_4	0	0	$-U_3$	$-U_7$
F_{66}'	$4U_5$	0	0	$-4U_3$	$-4U_7$
F_{16}'	0	$2U_6$	$-U_2$	$2U_7$	$-2U_3$
F_{26}'	0	$2U_6$	$-U_2$	$-2U_7$	$2U_3$
F_{13}'	V_1	V_2	$F_{36}/2$	0	0
F_{23}'	V_1	$-V_2$	$-F_{36}/2$	0	0
F_{36}'	0	F_{36}	$-2V_2$	0	0
F_{44}'	W_1	W_2	$-F_{45}$	0	0
F_{55}'	W_1	$-W_2$	F_{45}	0	0
F_{45}'	0	F_{45}	W_2	0	0
F_{33}'	F_{33}	0	0	0	0

(93)

where $U_1 = (3F_{11} + 3F_{22} + 2F_{12} + F_{66})/8$

$U_2 = (F_{11} - F_{22})/2$

$U_3 = (F_{11} + F_{22} - 2F_{12} - F_{66})/8$

$U_4 = (F_{11} + F_{22} + 6F_{12} - F_{66})/8$

$U_5 = (F_{11} + F_{22} - 2F_{12} + F_{66})/8$

$$U_6 = (F_{16} + F_{26})/4$$

$$U_7 = (F_{16} - F_{26})/4$$

$$V_1 = (F_{13} + F_{23})/2$$

$$V_2 = (F_{13} - F_{23})/2$$

$$W_1 = (F_{44} + F_{55})/2$$

$$W_2 = (F_{44} - F_{55})/2$$

The remaining 8 components of F_{ij} are those which appeared as zero in Equation (7b). They can also be expressed in terms of multiple angles. In the case of the rotation about the 3-axis, these components are sine and cosine functions of θ and 3θ. For odd-multiple angles, there are no constant terms and thus no invariants associated with these particular transformations. Since these components are not frequently encountered in the study of composites, we will not show them here.

The invariants associated with these transformations are T_1 and F_3 for F_i; and U_1, U_4, V_1, W_1 and F_{33} for F_{ij}. Invariant U_5 is not independent and it can be expressed by

$$4U_5 = 2(U_1 - U_4) \tag{94}$$

These invariants are arranged in matrix form in Equation (13).

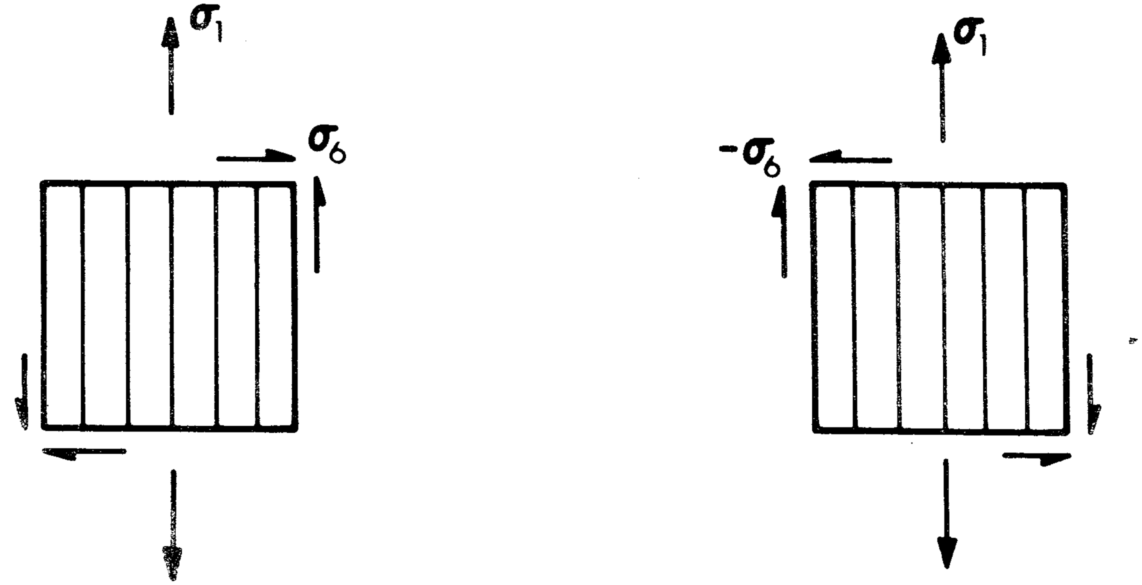

Fig. 1 - Strength of orthotropic material independent of sign of shear stress σ_6.

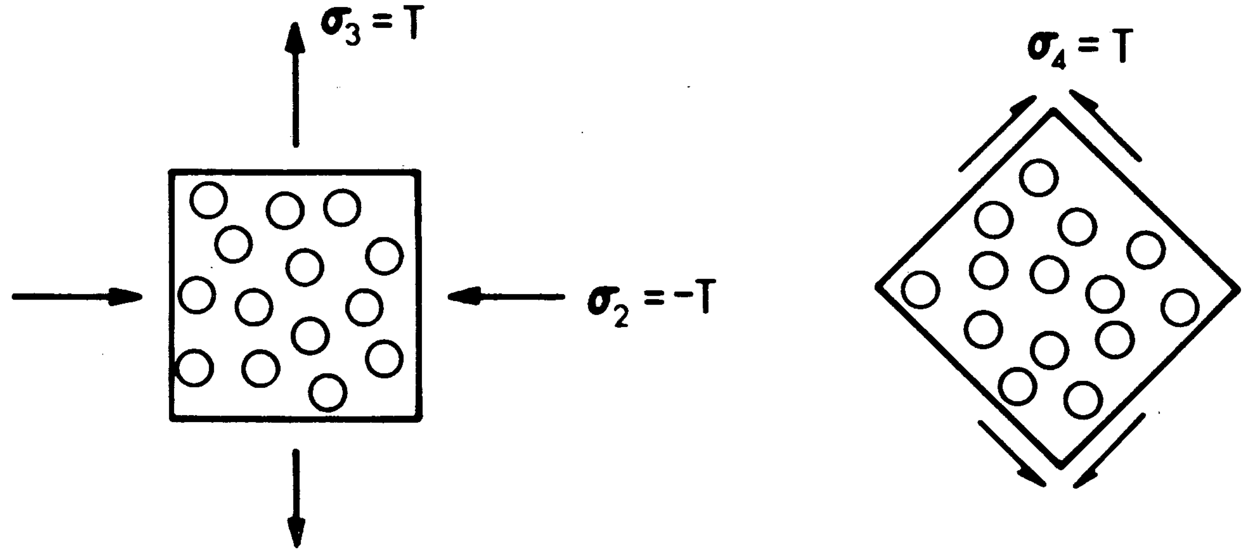

Fig. 2 - Equivalent states of stress in pure shear and tension-compression.

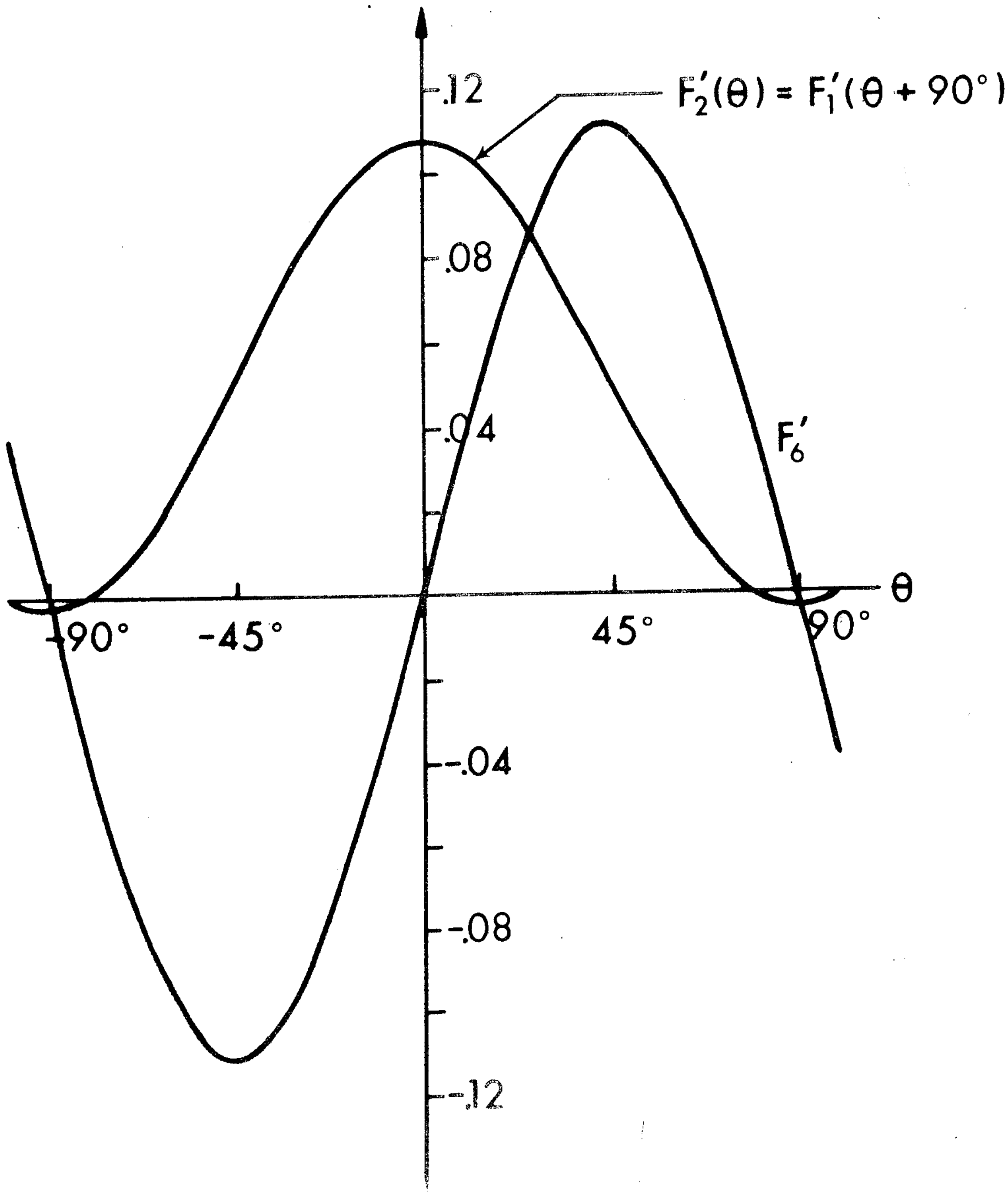

Fig. 3 – Transformation of strength tensor F_i for a graphite-epoxy composite system. Invariant T_1 represents the average value of the area under the F_1' curve.

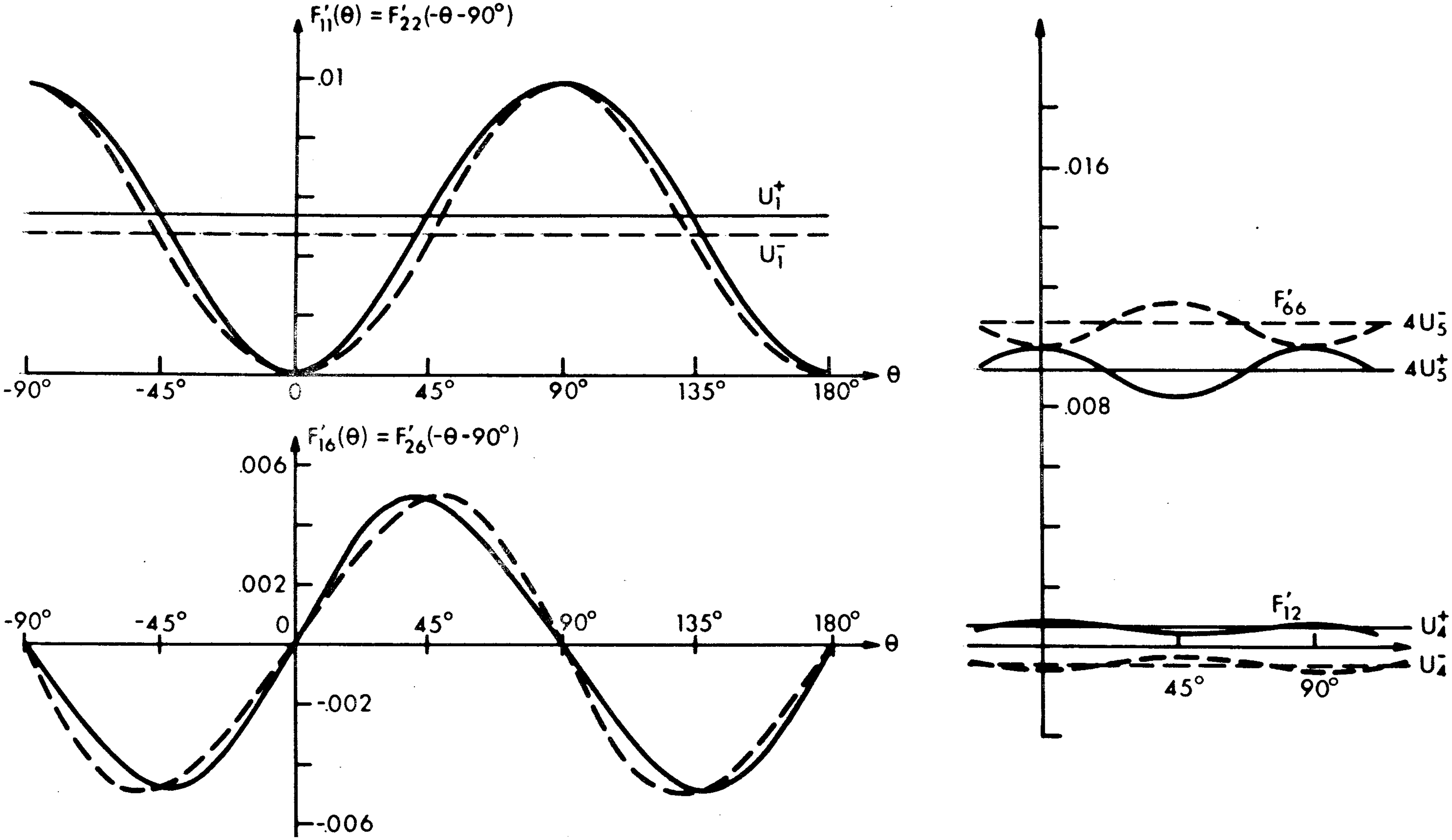

Fig. 4 – Transformation of F_{ij} for a graphite-epoxy composite. Solid lines represent the upper bound of F_{12}; dashed lines, the lower bound. Invariants associated with various components are shown as horizontal lines. The plus and minus superscripts on U_i correspond to the bounds of F_{12}.

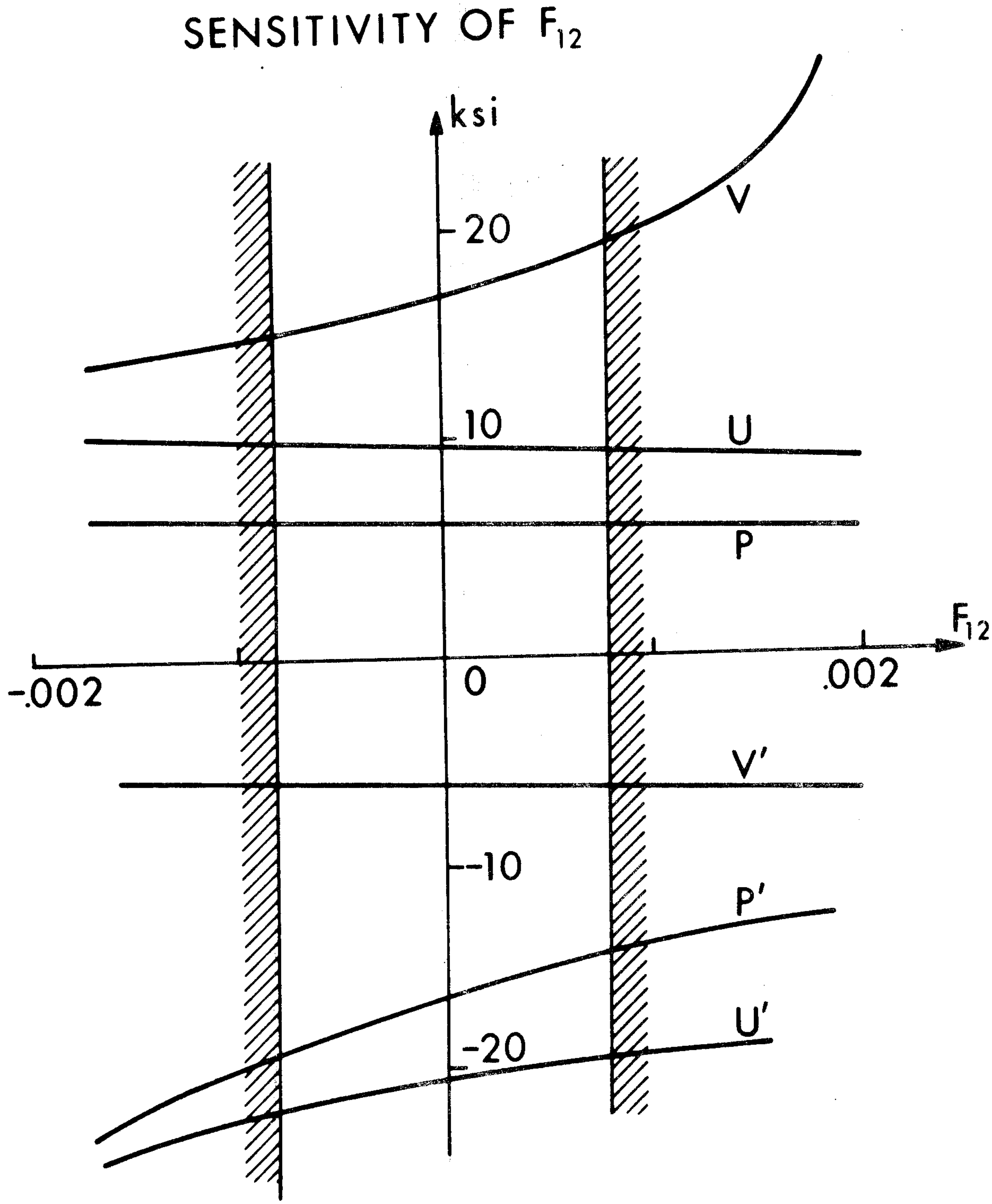

Fig. 5 - Effect of F_{12} on the combined-stress test data for graphite-epoxy composite. The bounds of F_{12} are shown. Quadratic approximations by Hill [3] and Hoffman [8] correspond to essentially zero value for F_{12} in this Figure.

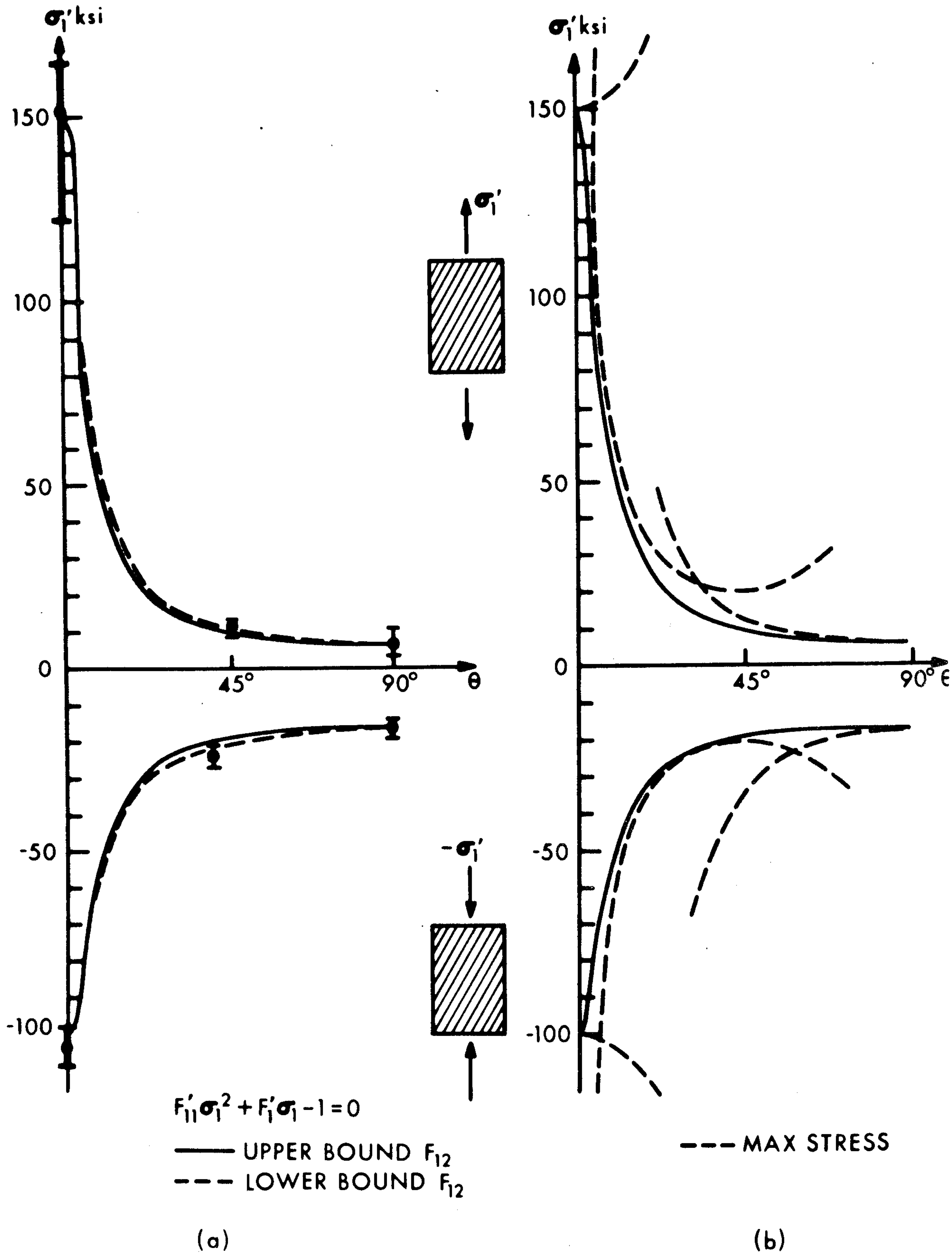

Fig. 6 - Off-axis uniaxial strength of graphite-epoxy composite

(a) Upper and lower bound of our theory and experimental data.

(b) Dashed lines, the maximum stress theory.

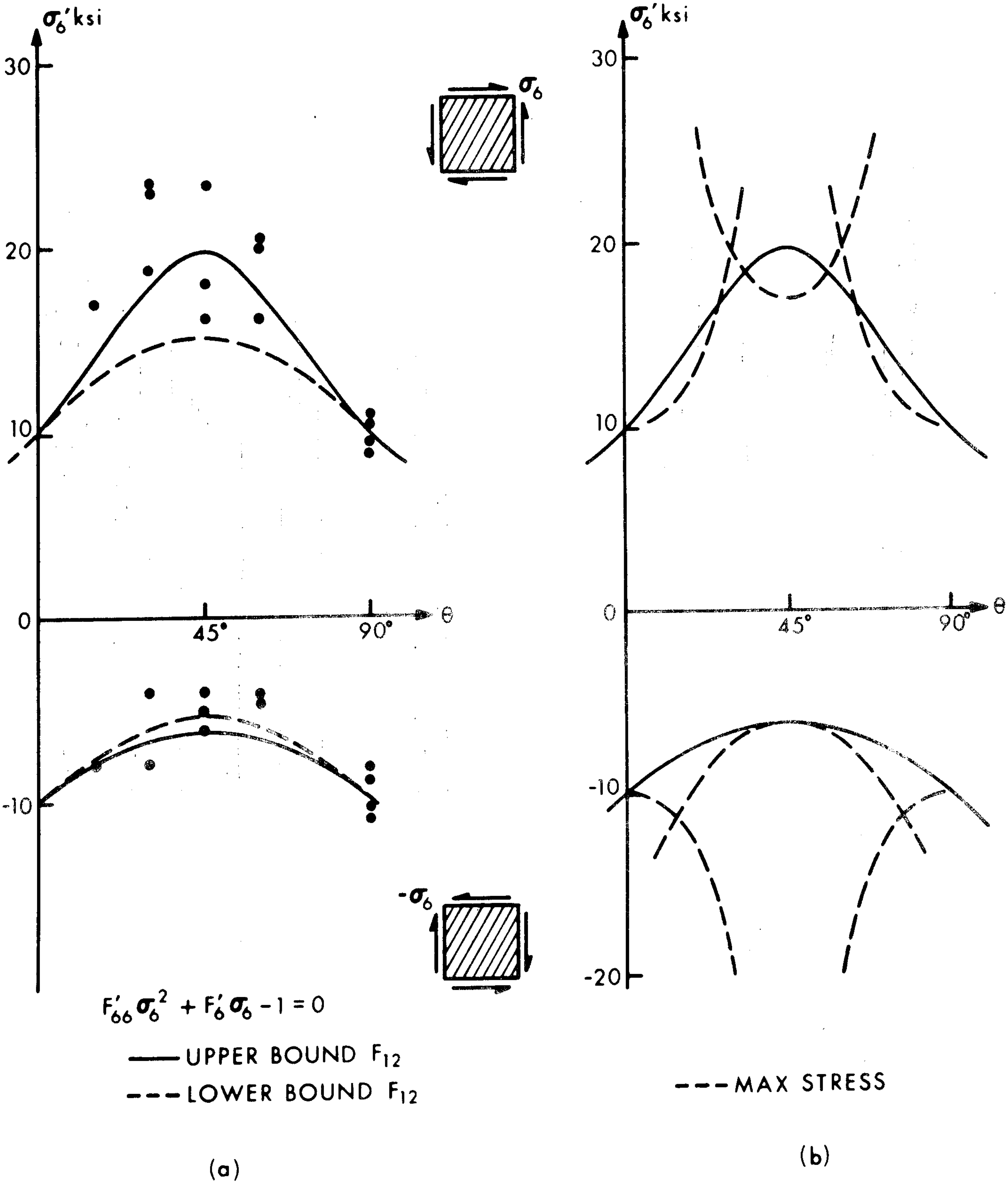

Fig. 7 - Off-axis shear strength of graphite-epoxy composite

(a) Upper and lower bound of our theory and experimental data from tubular specimens.

(b) Dashed lines, the maximum stress theory.

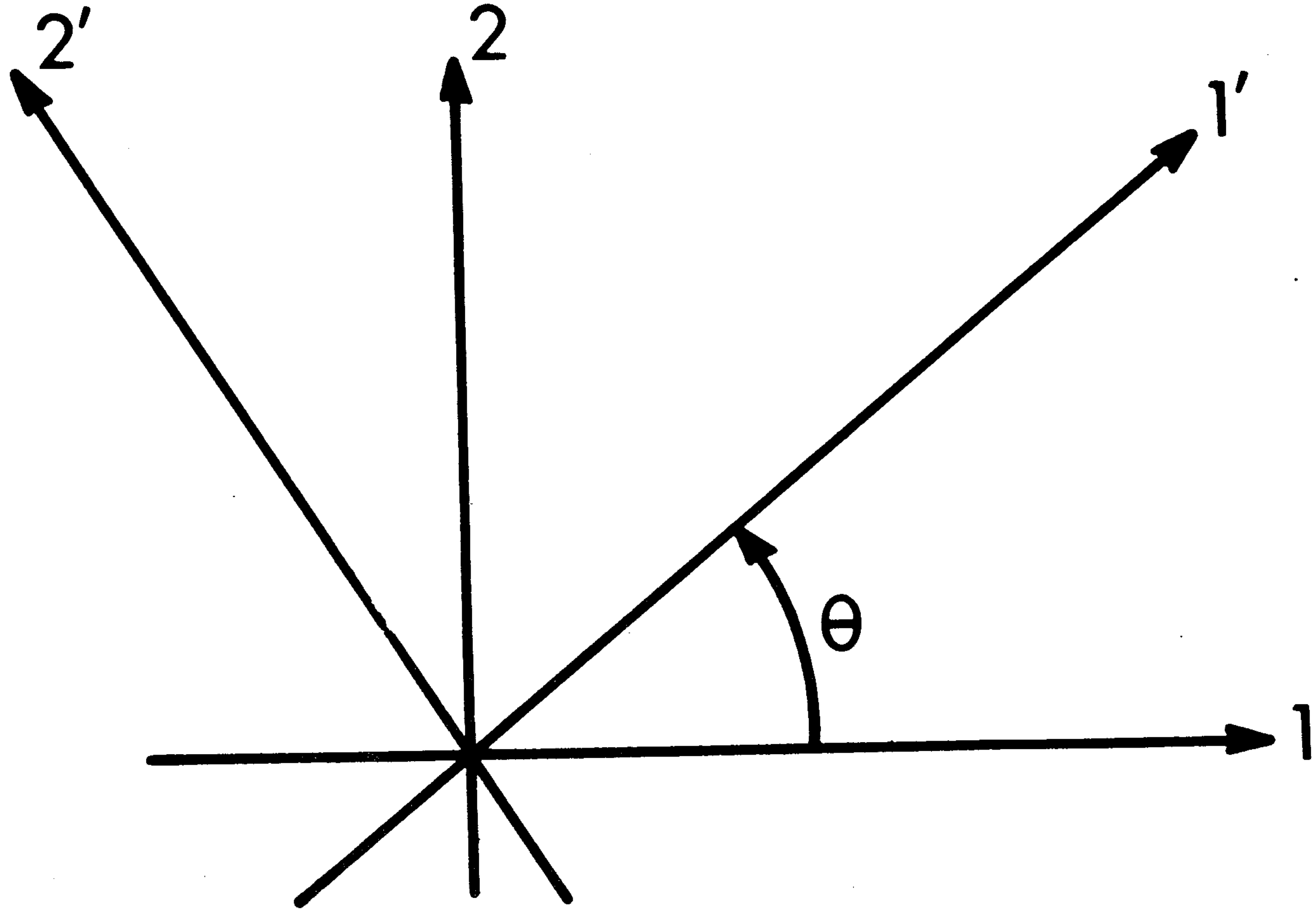

Fig. 8 - Direction of positive rotation of a coordinate transformation about the z- or 3-axis of a right-handed system.

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13. ABSTRACT

An operationally simple strength criterion for anisotropic materials is developed from a scalar function of two strength tensors. Differing from existing quadratic approximations of failure surfaces, the present theory satisfies the invariant requirements of coordinate transformation, treats interaction terms as independent components, takes into account the difference in strengths due to positive and negative stresses, and can be specialized to account for different material symmetries, multi-dimensional space, and multi-axial stresses. The measured off-axis uniaxial and pure shear data are shown to be in good agreement with the predicted values based on the present theory.

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